

How not to discretize the control

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Abstract

In this short note, we address the discretization of optimal control problems with higher order polynomials. We develop a *necessary and sufficient condition* to ensure that weak limits of discrete feasible controls are feasible for the original problem. We show by means of a simple counterexample that a naive discretization by higher order polynomials can lead to non-feasible limits of sequences of discrete solutions.

1 Introduction

We consider the discretization of the optimal control problem

$$\min_{u \in L^2(\Omega)} J(u) \quad \text{subject to } u \geq 0. \quad (1)$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded open set. We set $U := L^2(\Omega)$. The objective is given by $J : U \rightarrow \mathbb{R}$ and we assume that a (not necessarily unique) global solution \bar{u} of (1) exists. This can be guaranteed under standard assumptions on J . In particular, we have in mind to choose J as the reduced cost functional of an optimal control problem subject to a partial differential equation (PDE). For an introduction to optimal control problems for PDEs, we refer to [3]. In order to numerically solve the problem, it has to be discretized. We will investigate a particular choice of discretization, which consists of discretizing the controls on subdivisions of the domain Ω by, e.g., piecewise polynomial functions. Note that we do *not* address the related question “How to not discretize the control?”, which was popularized by Hinze [1].

2 Discretization

We consider a sequence of discretizations, indexed by an integer $n \in \mathbb{N}$. We associate with each n the following objects:

- (A1) a finite dimensional subspace $U_n \subset U$ with fixed basis $\{\phi_n^1, \dots, \phi_n^{N_n}\}$,
- (A2) a set \mathcal{T}_n of open, pairwise disjoint elements T with $\bar{\Omega} = \overline{\bigcup_{T \in \mathcal{T}_n} T}$ and $\text{diam}(T) \leq h_n$ for all $T \in \mathcal{T}_n$, where $h_n \searrow 0$,

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(A3) and a functional $J_n : U_n \rightarrow \mathbb{R}$ approximating J .

As a discretization for (1), we choose

$$\min J_n(u_n) \quad \text{subject to } u_n = \sum_{i=1}^{N_n} \lambda_i \phi_n^i, \quad \text{where } \lambda_i \geq 0 \quad \forall i = 1, \dots, N_n. \quad (2)$$

That is, the non-negativity constraint $u \geq 0$ is replaced by a non-negativity constraint on the coordinates of u_n with respect to the chosen basis of U_n . We assume that the discrete problem has a global solution \bar{u}_n . In order to study convergence with respect to $n \rightarrow \infty$ we will impose the following conditions on the sequence $\{(U_n, \mathcal{T}_n, J_n)\}_{n \in \mathbb{N}}$:

(A4) $J(u) \leq \liminf_{n \rightarrow \infty} J_n(u_n)$ for every sequence $\{u_n\}_{n \in \mathbb{N}}$ with $u_n \in U_n \forall n \in \mathbb{N}$ and $u_n \rightharpoonup u$ in U for $n \rightarrow \infty$.

(A5) $J(\bar{u}) \geq \limsup_{n \rightarrow \infty} J_n(v_n)$ for some sequence $\{v_n\}_{n \in \mathbb{N}}$ with $v_n \in U_n$.

Assumptions (A4), (A5) are slightly weaker than the Γ -convergence of J_n towards J . We remark that these assumptions are fulfilled for standard FE discretizations of optimal control problems subject to partial differential equations [2, 3].

3 Convergence of discrete approximations

In this section we will show that it is sufficient that the basis functions ϕ_n^i have non-negative integral on the cells $T \in \mathcal{T}_n$ to guarantee that weak limits of discrete solutions are feasible. Note, that non-negativity of the basis functions is not required.

Theorem 3.1. *Let the sequence $\{(U_n, \mathcal{T}_n, J_n)\}_{n \in \mathbb{N}}$ satisfy (A1)–(A5) and, in addition,*

$$\int_T \phi_n^i \geq 0 \quad \forall n \in \mathbb{N}, T \in \mathcal{T}_n, i = 1 \dots N_n^i.$$

Then, every weak limit u of feasible points u_n of (2) is feasible for (1), and the weak limit (if it exists) of global solutions \bar{u}_n of (2) is a global solution of (1).

Proof. Let the feasible points u_n of (2) converge weakly in U to u . We show that u is feasible for (1). Let $K \subset \Omega$ be compact and non-empty. We set

$$K_n := \bigcup_{T \in \mathcal{T}_n: \bar{T} \cap K \neq \emptyset} \bar{T}.$$

Then $\int_{K_n} u_n \geq 0$ for all $n \in \mathbb{N}$. By dominated convergence and condition (A2), $\chi_{K_n} \rightarrow \chi_K$ in $L^2(\Omega)$, which implies $\int_{K_n} u_n \rightarrow \int_K u$. Hence $\int_K u \geq 0$ for all such compact K , and a density argument implies $u \geq 0$.

Now, let \bar{u}_n be globally optimal for (2) and denote by \tilde{u} the weak limit. As in the first part of the proof, we can show $\tilde{u} \geq 0$. Due to (A4), (A5), we have $J(\tilde{u}) \leq \liminf_{n \rightarrow \infty} J_n(\bar{u}_n) \leq \liminf_{n \rightarrow \infty} J_n(v_n) \leq \limsup_{n \rightarrow \infty} J_n(v_n) \leq J(\bar{u})$. Since \tilde{u} is feasible for (1), and \bar{u} is a global solution, it follows that \tilde{u} is a global solution. \square

4 An example with non-feasible limit

We consider the following simple optimal control problem:

$$\min_{(y,u) \in H^1(\Omega) \times L^2(\Omega)} \|y - y_d\|_{L^2(\Omega)}^2 + \alpha \|u\|_{L^2(\Omega)}^2$$

subject to

$$u \geq 0$$

and

$$\int_{\Omega} \nabla y \cdot \nabla v + y v = \int_{\Omega} u v \quad \forall v \in H^1(\Omega).$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded domain with polygonal boundary. We set $y_d := -1$ and $\alpha > 0$. It is easy to check that $(\bar{y}, \bar{u}) = (0, 0)$ with $J(\bar{u}) = |\Omega|$ is the unique solution of this problem: due to the maximum principle it holds $y \geq 0$ for all feasible pairs (y, u) .

We discretize this problem by finite elements on simplicial decompositions \mathcal{T}_n of Ω . As finite element spaces we choose standard (discontinuous or continuous) Lagrange elements of polynomial degree $k \geq 1$. Further, we define $J_n(u_n) := \|y_n(u_n) - y_d\|_{L^2(\Omega)}^2 + \alpha \|u_n\|_{L^2(\Omega)}^2$, where $y_n(u_n)$ is the solution of a suitably discretized state equation. Following standard arguments [2] it is easy to prove that **(A1)**–**(A5)** are satisfied.

We denote by $(\hat{\psi}_1 \dots \hat{\psi}_m)$ the Lagrange basis of order k on the reference simplex \hat{T} . We will show that if there is a basis function with negative integral, then the solutions of the discretized problem will converge weakly to a non-feasible limit u .

Theorem 4.1. *Assume that $\int_{\hat{T}} \hat{\psi}_j < 0$ for some $j \in \{1, \dots, m\}$. Denote by \bar{u}_n the unique solution of the discretized problem. Then it holds $\bar{u}_n \rightharpoonup u$ (along a subsequence), where u does not satisfy $u \geq 0$.*

Proof. We define $I_n := \{i : \int_{\Omega} \phi_n^i < 0\}$. Due to the assumption this set is non-empty. We set $w_n := \sum_{i \in I_n} \phi_n^i \in U_n$. Then for each element $T \in \mathcal{T}_n$ the function $w_n|_T$ is an affine transformation of $\hat{w} := \sum_{j: \int_{\hat{T}} \hat{\psi}_j < 0} \hat{\psi}_j$. Consequently, it holds $\|w_n\|_{L^2(\Omega)}^2 = \sum_{T \in \mathcal{T}_n} |T| |\hat{K}|^{-1} \|\hat{w}\|_{L^2(\hat{K})}^2 = |\Omega| |\hat{K}|^{-1} \|\hat{w}\|_{L^2(\hat{K})}^2 =: M^2$, which shows that $\{w_n\}_{n \in \mathbb{N}}$ is uniformly bounded. In addition, it holds

$$\int_{\Omega} w_n = \sum_{T \in \mathcal{T}_n} |T| |\hat{K}|^{-1} \int_{\hat{K}} \hat{w} = |\Omega| |\hat{K}|^{-1} \int_{\hat{K}} \hat{w} =: -\beta < 0.$$

We set $z_n := y_n(w_n)$. Testing the discretized equation by z_n and 1 yields $L_n := \|z_n\|_{L^2(\Omega)} \leq \|w_n\|_{L^2(\Omega)}$ and $\int_{\Omega} z_n = \int_{\Omega} w_n = -\beta < 0$, respectively.

Then it holds $J_n(t w_n) = (t^2 L_n^2 - 2 t \beta + |\Omega|) + \alpha t^2 M^2 \leq (1 + \alpha) M^2 t^2 - 2 t \beta + |\Omega|$. For the choices $\hat{t} := \frac{\beta}{(1+\alpha) M^2} > 0$ and $u_n := \hat{t} w_n$ we obtain $J_n(u_n) \leq |\Omega| - \delta < |\Omega| = J(\bar{u})$ with $\delta := \beta \hat{t} > 0$. This shows that $J_n(\bar{u}_n) \leq |\Omega| - \delta < J(\bar{u})$.

Let $\bar{u}_n \rightharpoonup u$ in $L^2(\Omega)$ (along a subsequence) and, consequently, $y_n(\bar{u}_n) \rightharpoonup y$ in $H^1(\Omega)$. By standard arguments, (y, u) satisfy the weak formulation of the partial differential equation. Moreover, as in the proof of Theorem 3.1 it follows $J(u) \leq |\Omega| - \delta < |\Omega| = J(\bar{u})$. This implies that (y, u) cannot be feasible, and consequently $u \geq 0$ is violated. \square

Numerical experiments show that basis functions with negative integral appear for sufficiently large k depending on the spatial dimension. For the standard Lagrangian elements, we found the following situation:

$d = 1$: basis function have non-negative integrals for $k \in \{1, 2, 3, 4, 5, 6, 7, 9\}$,
but not for $k \in \{8, 10, 11\}$,

$d = 2$: basis function have non-negative integrals for $k \in \{1, 2, 3, 5\}$,
but not for $k \in \{4, 6, 7, 8\}$,

$d = 3$: basis function have non-negative integrals for $k \in \{1, 3\}$,
but not for $k \in \{2, 4, 5, 6\}$.

Here, in particular the situation in dimension 3 is remarkable: A naive control discretization by P^2 elements with non-negativity constraints on the coefficients may fail to produce approximations with feasible limits!

Let us remark that similar results can be proven for problems with homogeneous Dirichlet boundary conditions and for more general discretizations using an affine family of finite elements.

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